How to code the simplest « Plant model » in MatCont

What you will have to learn (during the atelier) is the following:

1. Define a model in MatCont;
2. Compute simple solutions (orbits) to explore the dynamics and to find equilibria;
3. Using the steady states found this way, you will use MatCont to "continue" (=trace) the equilibrium through parameter space, i.e., to draw the curve of the equilibrium as a function of one parameter;
4. MatCont will detect bifurcations on such "equilibrium curves". You will learn how to "continue" (=trace) the bifurcations through parameter space. That is, draw the location of the bifurcation as a function of two (or more) parameters.

Below you will find a "cook book" explanation of how to implement a model in MatCont, and how to analyse it. Of course this is only a first step; this description is by no means complete. Many features of MatCont are not explained here, and can be explained "live" by the teachers, or can be found in the tutorial or user manual.

You will also find some excercises to do on paper with a pen or pencil. Please do these analyses, they are extremely important for a successful implementation & understanding of the numerical results!!

You will also find some questions. Please answer all of them carefully.

Below we show how to implement a model that is based on Figures 1(a) and 1(b) of the review article by Van de Koppel et al (1997). The panels (a) and (b) of the figure 1 are copied below:

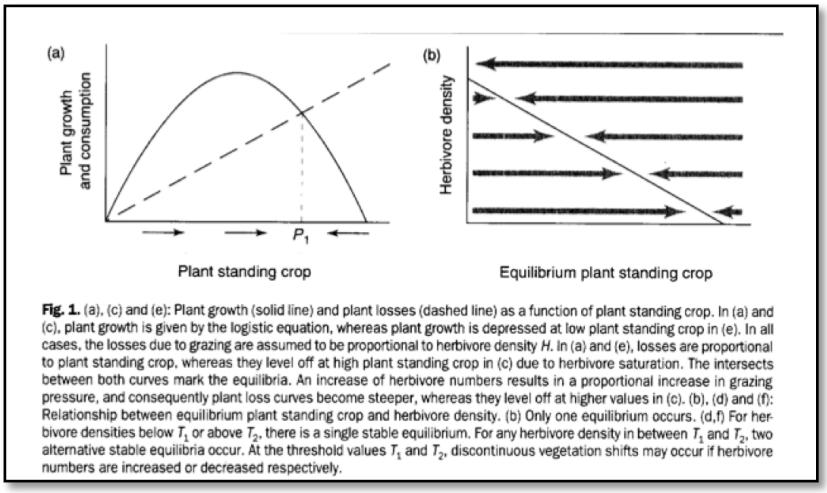
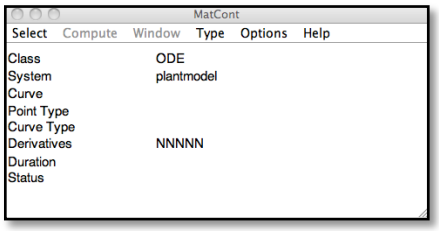


Fig. 1. (a), (c) and (e): Plant growth (solid line) and plant losses (dashed line) as a function of plant standing crop. In (a) and (c), plant growth is given by the logistic equation, whereas plant growth is depressed at low plant standing crop in (e). In all cases, the losses due to grazing are assumed to be proportional to herbivore density *H*. In (a) and (e), losses are proportional to plant standing crop, whereas they level off at high plant standing crop in (c) due to herbivore saturation. The intersects between both curves mark the equilibria. An increase of herbivore numbers results in a proportional increase in grazing pressure, and consequently plant loss curves become steeper, whereas they level off at higher values in (c). (b), (d) and (f): Relationship between equilibrium plant standing crop and herbivore density. (b) Only one equilibrium occurs. (d,f) For herbivore densities below T_1 or above T_2, there is a single stable equilibrium. For any herbivore density in between T_1 and T_2, two alternative stable equilibria occur. At the threshold values T_1 and T_2, discontinuous vegetation shifts may occur if herbivore numbers are increased or decreased respectively.

The "cook book" description of implementing this model

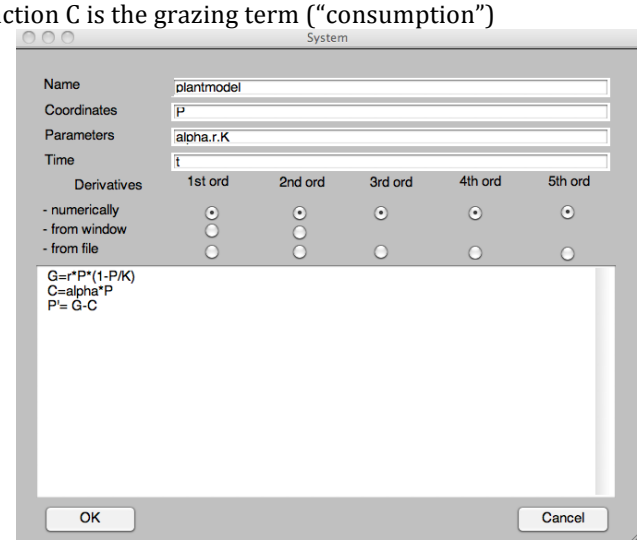
- Open Matlab and MatCont. This is the **MatCont** window:



  You can "minimize" the Matlab window (which you do not need), but not the MatCont window.

- In the MatCont window, choose Select – System – New
- In the **System** window, define the model in the following way:
  - P is plant density
  - r is the exponential growth rate of plants (at low plant density)

- K is the carrying capacity of plants
- alpha is the grazing intensity
- The function G is the growth term
- The function C is the grazing term ("consumption")



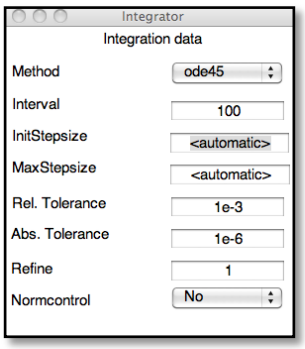
- Hints: coordinates = variables. Use commas in between parameters and coordinates, no spaces.
- You can define a "function" just by typing "A=a*b" where A is your function name, and a*b would be the function of parameters, coordinates and other functions. For example, see the functions G and C above.
- P' means the derivative dP/dt

- DO THIS WITH PEN & PAPER:
- Write down the differential equation for P.
- *Give the expressions for the two possible eqiulibria.*
- *Derive the condition, in terms of the parameters alpha, r and K, for which the system has either a single (positive) equilibrium, or two.*

- Note that the Select – System menu can also be used to open ("Load") existing model definitions and to modify them.

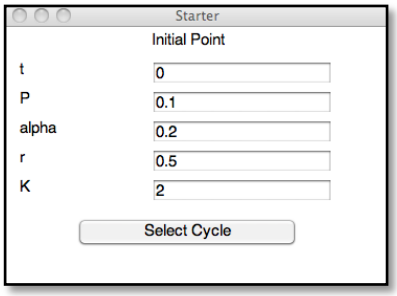
- Now you are ready to start computing. Of course, to integrate the equation you need to give numerical values to the parameters and the initial conditions. First, indicate the type of computation, by choosing a type of initial point (Type – Initial point – Point) and the type of curve to be computed. As a rule, the first thing to do is a simple "Orbit", starting from an ordinary "Point". Note that the MatCont window now indicates point type and curve type. The notation used is P_O(1), meaning Orbit number "1", starting from a Point.

- The **Integrator** window



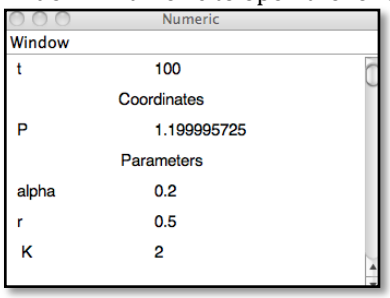
- This window allows you to choose the integration method, the integration interval, step size etc. Change the interval to 100. It is a good idea to leave the other values as they are (unless you have good reason to change them).

- The **Starter** window



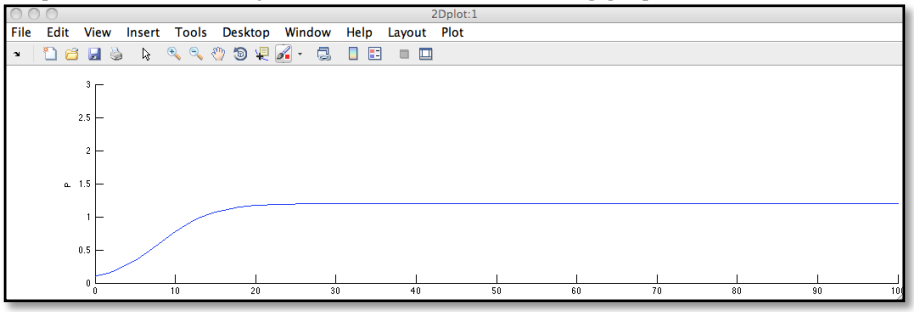
  - This window allows you to specify the parameter values and the initial conditions of the variables (t and P) for the integration (the orbit).
  - Do not click **Select Cycle** (we do not use that feature now)

- The **Numeric** window
  - Choose Window – Numeric to open the following numeric window:



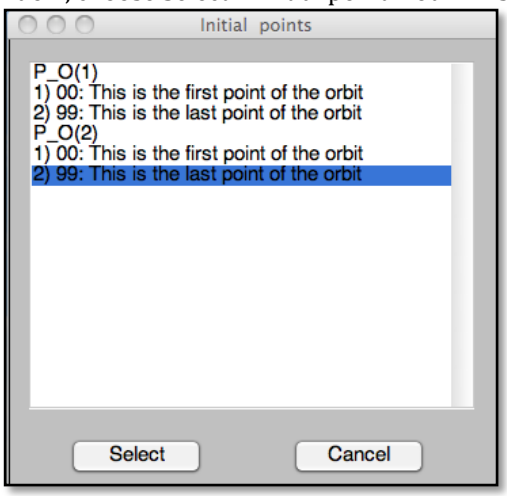
  This window displays the values of variables, parameters, eigenvalues etc during your computations. Choose Window – Layout to determine the layout (which variables etc to display in the table). NB this window can be very useful during continuation curves (see below) because you can display the **eigenvalues** during a computation, which will tell you the stability of the current point.

- Now you can start a computation by clicking Compute – Forward in the MatCont window.
- Next you want to see the graph of the orbit.
  - First, on the MatCont window, choose a "Window – Graphic – 2Dplot" .
  - put time on the Abscissa, P on the Ordinate. Use the menu "Time, Coordinates, Parameters" to find t and P.
    - NB You can change this at any time by choosing on the **2Dplot window** the menu Layout – Variables on axes.
  - Then adjust the axes, by choosing Layout – Plotting region. Choose 0-100 for the abscissa and 0-3 for the ordinate.
  - Compute the orbit, and you should have the following graph:



- You can explore the model by running orbits for different parameters & initial conditions.

- Now you can start with the **"continuation" of the found equilibrium**.
  In the MatCont window, choose Select – Initial point. You will see the Initial points window:

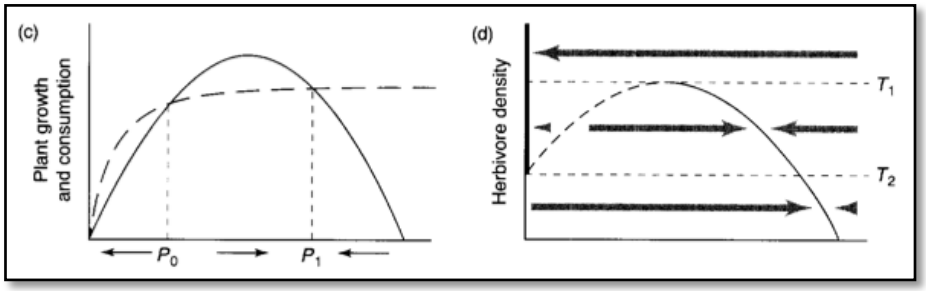


  - This lists the two orbits I have run, P_O(1) and P_O(1), and the "special points" that are associated with these orbits, in this case, the first and last points of the orbits.
  - Select the last point of the last orbit, that corresponds to the equilibrium of the current parameter settings.
  - Note the changes in the Starter window; it now contains the values of the last point of the orbit (do not click Select Cycle).
  - Select Type – Initial point – Equilibrium
  - Select Type – Curve – Equilibrium
  - The **Starter** window has changed: it now lets you choose the parameter over which to continue the equilibrium. Choose one of them.
  - The **Continuer** window has appeared. It contains the values for the parameters used during the continuation process. You can leave the values as they are.

- Open a new 2Dplot and draw the equilibrium curve over alpha. Compute both forward and backward, if necessary, to have the full equilibrium curve.
  - *What is the plant density in the absence of grazing?*
  - *At what grazing level does the plant population go extinct?*
- MatCont indicates "BP" at the extinction point. This means "Branch point", which is the same thing as a transcritical bifurcation.
- Choose the BP as initial point and draw the equilibrium curve (NB be careful to choose the correct point and curve types!).
  - *Which equilibrium are you now continueing?*

Model extension: saturation of consumers (Holling's type II functional response)

Consider the following modification of the (graphical) model:



The new grazing function can be modeled as follows:

C=alpha*P/(1 + alpha*beta*P)

- DO THIS WITH PEN & PAPER:
- Write down the differential equation for P
- Analyse the model:
- *Sketch the functions C and G for the cases with one, two and three equilibria.*
- *(Only if you like algebra: Find expressions for the three equilibria.)*
- *Under what approximate conditions (in terms of the parameters alpha, beta, r and K* ***– note that you may approximate****) has the model either one, two or three equilibria?*
- *In case of three equilibria, which ones are stable/unstable?*

- Implement the modified model in MatCont
- Using the values r=0.5, K=2, alpha=1, beta=6, run orbits with the model, using initial conditions P=2, P=1, P=0.5, P=0.4, P=0.3, P=0.2, P=0.1
    - Does this correspond to you answers to the questions above (pen & paper)?
- Continue the equilibrium curve using beta as your bifurcation parameter. Remember your epectations based on the pen & paper exercise!!
    - *Does the result conform to the mathematical analysis?*
- LP = Limit point. In MatCont, this refers to a saddle-node bifurcation.
- Try to continue the LP bifurcation in the alpha-beta parameter space.

A model of forest & spruce budworm dynamics

- Read carefully the text below.
- Implement the ***scaled model*** in MatCont (ie, the model of the dynamics of the scaled variable *b*, not *B*. The scaled model has only two parameters, R and Q).
- Explore its dynamics.
- Draw the bifurcation diagram., referred to as "Fig 3" in the text below.
- Indicate R1 and R2 in your bifurcation diagram. What are their values?
- Read the section on this model on the article by Ludwig et al (1997). How can you modify the model in order to account explicitly for the tree dynamics?

From: Ludwig, D., B. Walker, and C. S. Holling. 1997. Sustainability, stability, and resilience. Conservation Ecology [online]1(1): 7

"The following model was used by Ludwig, Jones, and Holling (1978) to understand the dynamics of the spruce budworm. The quantity *B* represents budworm density, measured in larvae per acre. This density is assumed to vary in time according to

$$\frac{dB}{dt} = r_B B\left(1 - \frac{B}{K_B}\right) - \beta\frac{B^2}{\alpha^2 + B^2}, \qquad (14)$$

where r_B is an intrinsic growth rate at low densities, K_B is a carrying capacity for the budworm in the absence of predation, and the second term in Equation (14) is a predation rate. The predators are assumed to have a Holling type-III functional response, with a maximum predation rate of β and a half-saturation budworm density of α. This functional form implies that predators have their greatest influence upon dynamics at intermediate ranges of budworm densities. At low densities, the predators search for alternate prey, because returns from foraging for budworm are relatively low. At high densities, budworms swamp their predators; thus, the predators have a small per capita effect, just as predators have a small per capita effect on large schools of fish. The parameter α is proportional to a measure of foliage density, because the predators search foliage for the budworms and their response is mediated by the number of budworms per unit of foliage. Hence, α is actually a state variable that generally changes on a slower time scale than the budworm. For the moment, we regard α as a constant.

Some algebra supplied in Ludwig et al. (1978) shows that there are either two or four equilibria for the budworm, depending upon the sizes of the dimensionless parameters *R* and *Q*, given by

$$R = \frac{r_B\alpha}{\beta}, \quad Q = \frac{K_B}{\alpha}. \qquad (15)$$

These equilibria satisfy

db/dt =

$$Rb\left(1 - \frac{b}{Q}\right) - \frac{b^2}{1 + b^2} = 0, \qquad (16)$$

where *b*=*B*/alpha. The equilibrium b=0 is always unstable, because db/dt > 0 if *b* is small and positive. The highest equilibrium is always stable, because db/dt < 0 if *b* is very large and positive. Thus, if there are only two equilibria, budworm density always moves toward the upper equilibrium. When there are four equilibria, they alternate in stability. A typical case is shown in "Fig. 3". If *R* is between R_1 and R_2, *b* may approach either the high equilibrium or the low equilibrium, depending upon whether the starting position of *b* is above or below the unstable equilibrium, which is the *separatrix*."