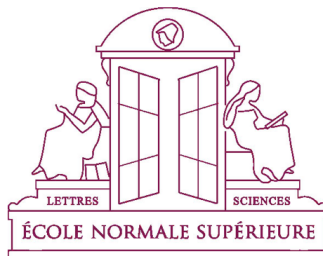


What is a Tipping Point & Why Do We Care?

Michael Ghil

**Ecole Normale Supérieure, Paris, and
University of California, Los Angeles**




Prière de visiter ces sites pour plus ample info.

<http://www.atmos.ucla.edu/tcd/>

<http://www.environnement.ens.fr/>

Motivation

- There's a lot of talk about "tipping points."
- It sounds threatening, like falling off a cliff: that's why we care!
- But what are they, and what do we know about them?
- Here's a disambiguation page (cf. Wikipedia), first.
- **Sociology**: "the moment of critical mass, the threshold, the boiling point" (Gladwell, 2000); a previously rare phenomenon becomes rapidly and dramatically more common.
- **Physics**: the point at which a system changes from a stable equilibrium into a new, qualitatively dissimilar equilibrium (throwing a switch, tilting a plank, boiling water, etc.).
- **Climatology**: "A climate tipping point is a somewhat ill-defined concept [...]" — so we'll try to actually define it better.

- **Catastrophe theory**: branch of bifurcation theory in the study of dynamical systems; here, a tipping point is "a parameter value at which the set of equilibria abruptly change." → **Let's see!**

M. Gladwell (2000) *The Tipping Point: How Little Things Can Make a Big Difference*.

T. M. Lenton *et al.* (2008) Tipping elements in the Earth's climate system, *PNAS*, v. **105**.

Approaching a state shift in Earth's biosphere

Anthony D. Barnosky^{1,2,3}, Elizabeth A. M. Getz⁴, John Harte^{5,10}, Alan Geerat Vermeij⁶, John W. Williams⁷, David P. Mindell⁸, Eloy Revilla⁹ & A. J. Valleron¹¹

Localized ecological systems are likely to approach critical thresholds. Here we show that as the Earth approaches a planetary-scale tipping point, the transitions on global as well as local scales necessary to address root causes

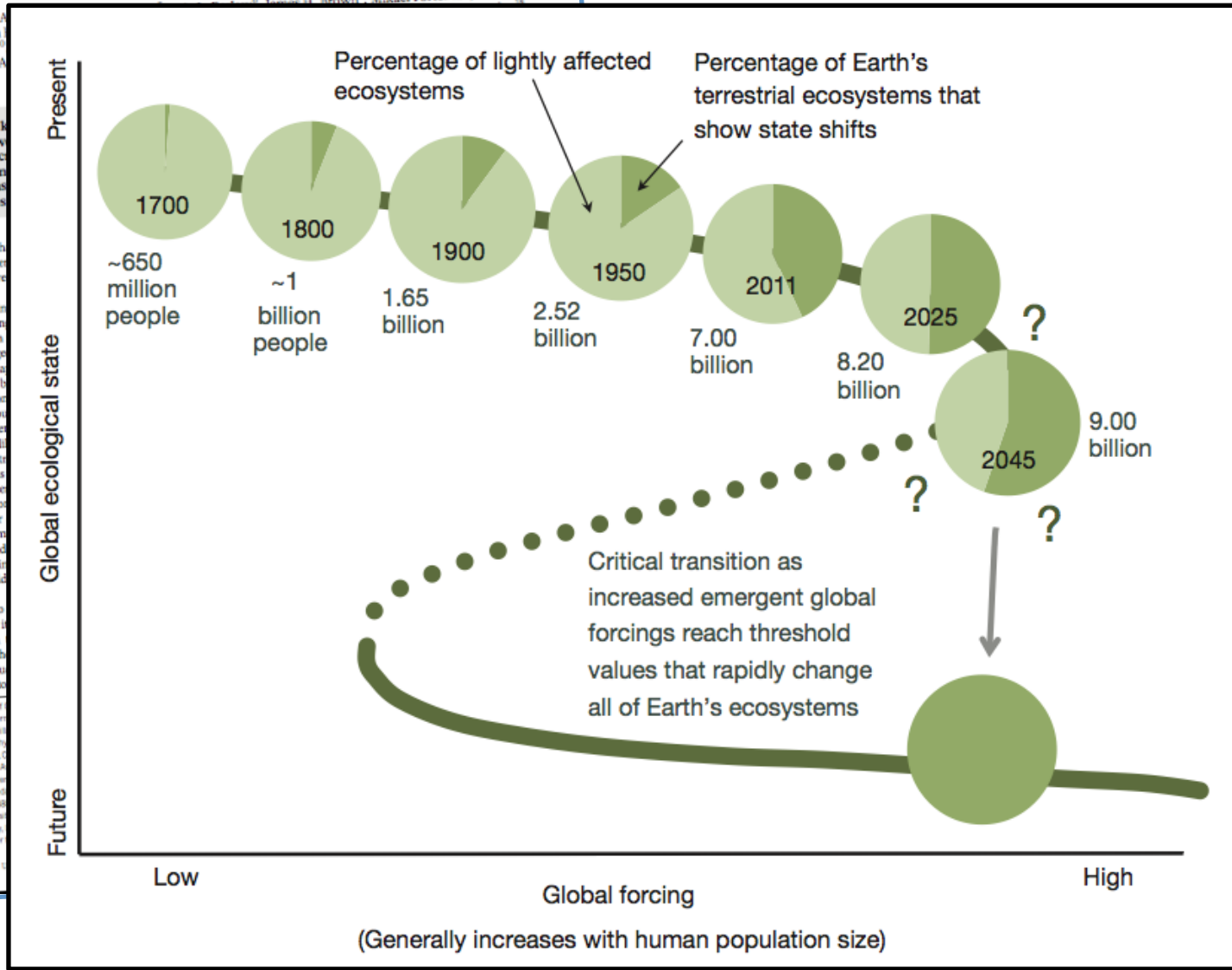
Humans now dominate Earth, and their activities have led to a growing interest in forecasting trends into the future assuming

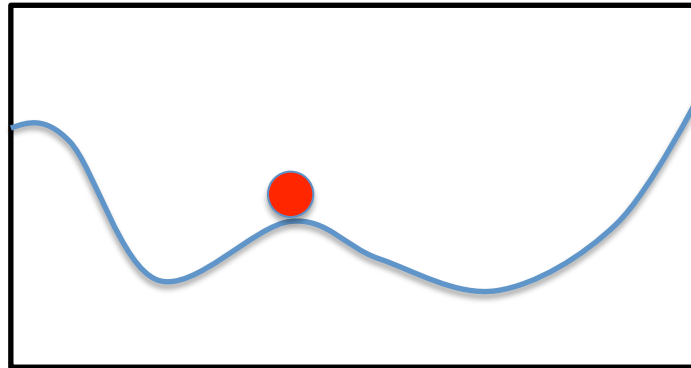
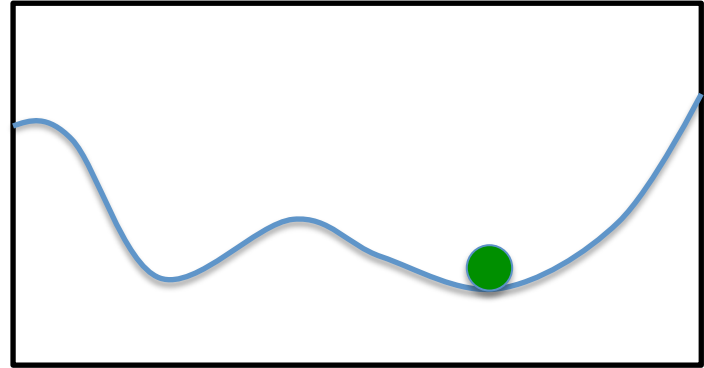
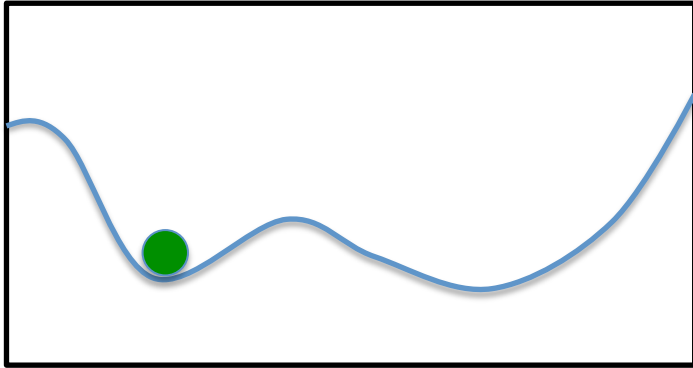
However, most biological forecasting recent trends into the future assuming species distribution changes may alter presently observed ge

Particularly important are recent decisions' caused by threshold effects are likely to be abrupt and irreversible

Two conclusions emerge. First, to avoid adversely impact humanity, it is essential to anticipate critical planetary scale and understanding biological changes. Second, as was also concluded, global-scale state shift, or at least to

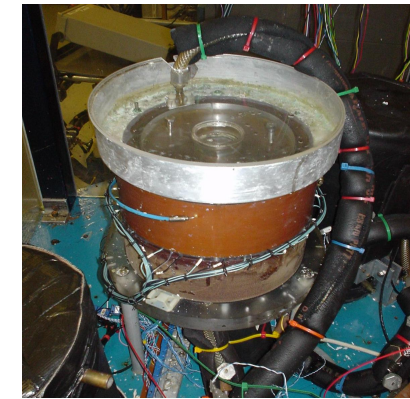
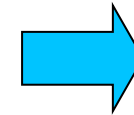
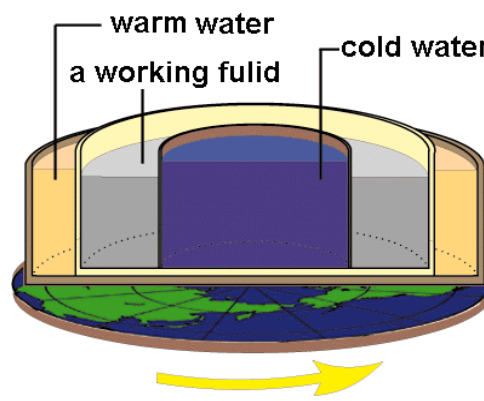
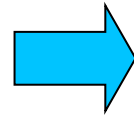
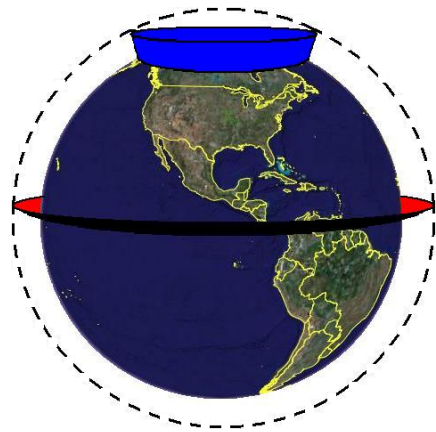
¹Department of Integrative Biology, University of California, Berkeley, California, USA; ²Department of Geosciences and Geography, University of California, Berkeley, California, USA; ³Department of Geology, University of California, Berkeley, California, USA; ⁴Department of Geology, University of California, Berkeley, California, USA; ⁵Instituto de Ecología y Biodiversidad, Pontificia Universidad Católica de Chile, Alameda 3401, Santiago, Chile; ⁶Department of Biological Sciences, Simon Fraser University, 88-111, Burnaby, British Columbia, Canada; ⁷Department of Geology, University of California, Berkeley, California, USA; ⁸Department of Biology, University of California, Berkeley, California, USA; ⁹Department of Biology, University of California, Berkeley, California, USA; ¹⁰Department of Geology, University of California, Berkeley, California, USA; ¹¹Center for Global Change Science, University of California, Berkeley, California, USA





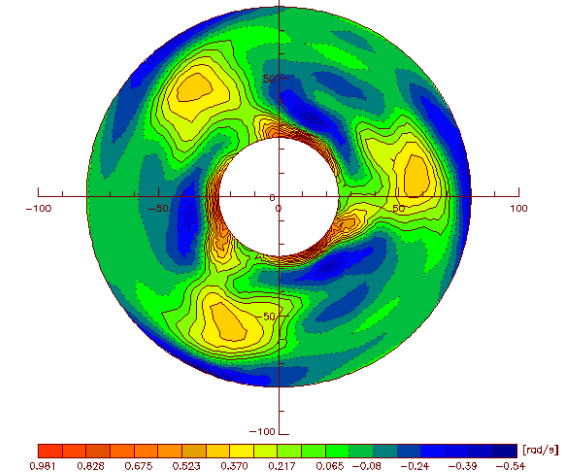
Unstable state

Laboratory Analogues of Planetary Atmospheric Circulation Systems



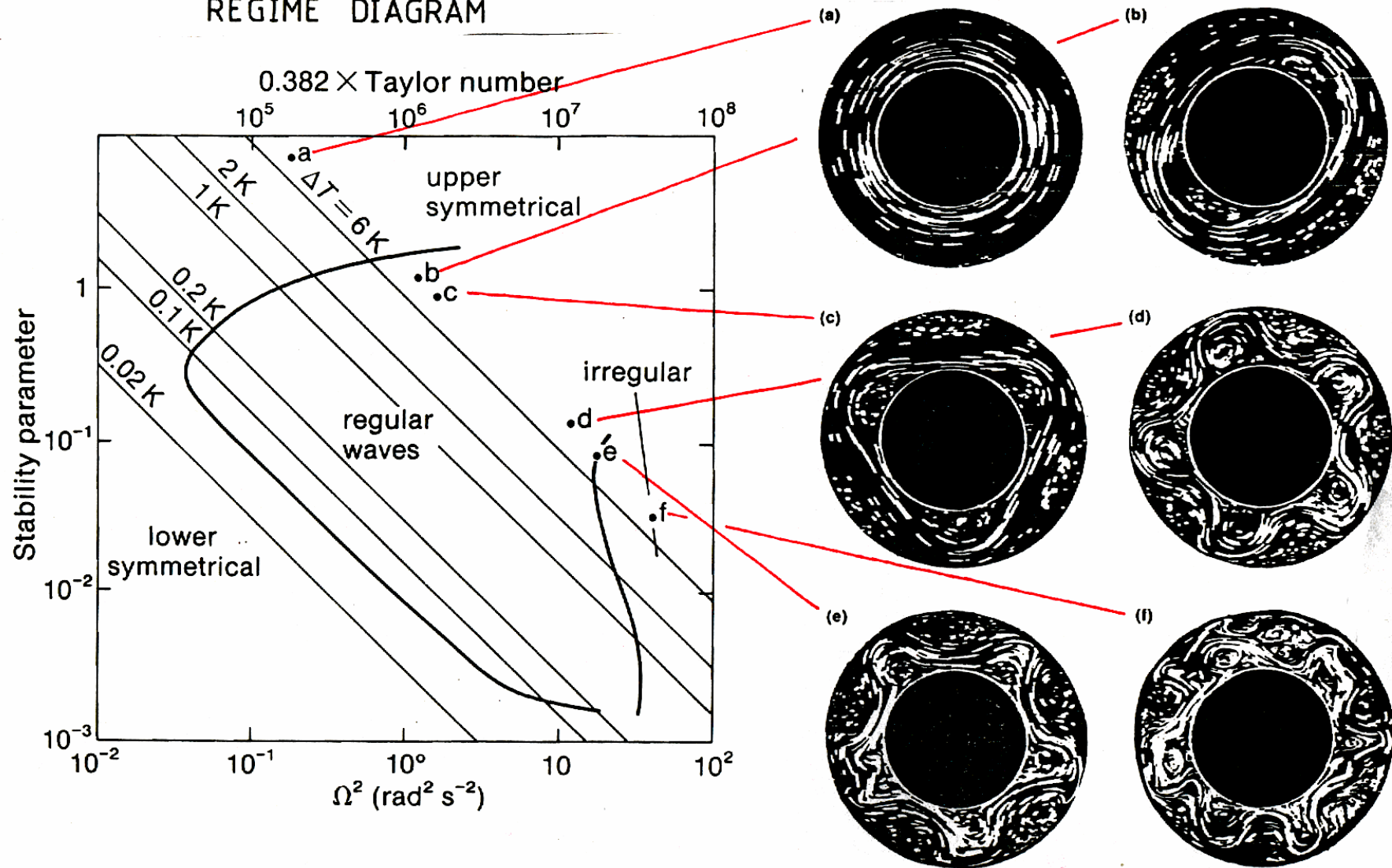
Const. $\Delta T = 4.01^\circ\text{C}$ Level 1
Relative vorticity (vert. comp.) exp1/101
Time Interval: 2000-2025 s

- *Baroclinic instability*
- a potential energy releasing instability in the atmosphere and oceans



Rotating Convection: An Illustration

TYPICAL FLOWS AND
REGIME DIAGRAM



Théorie élémentaire des bifurcations et *principe variationnel*

- **Points fixes**
 - stabilité linéaire
 - stabilité non linéaire et bassins d'attraction
- **Bifurcations de col**
 - branches multiples de solutions stationnaires
 - stabilité linéaire
- **Bifurcations en 1-D**
- **Stabilité non linéaire et principe variationnel**
 - principe variationnel en 0-D
 - principe variationnel en 1-D
- **Bistabilité et hystérésis**

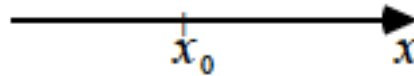
1. Fixed points

We start with a **scalar** ordinary differential equation (ODE)

$$\dot{x} = f(x; \mu),$$

depending on the parameter μ .

Linear stability, $\mu = 1$.



$$f(x_0) = 0 \Rightarrow \dot{x} = 0 \Rightarrow x \equiv x_0 \text{ — Fixed point (FP).}$$

Consider an initial perturbation at $t = 0$:

$$x(0) = x_0 + \xi(0), \tag{1}$$

$$\dot{x} = \dot{x}_0 + \dot{\xi} = \dot{\xi}, \tag{2}$$

$$= f(x_0 + \xi) = f(x_0) + f'(x_0)\xi + \mathcal{O}(\xi^2). \tag{3}$$

For an infinitesimal perturbation $\xi(0) = \xi_0$:

$$\begin{aligned} \dot{\xi} &= f'(x_0)\xi, & f'(x_0) &= \lambda, & \dot{\xi} &= \lambda\xi, \\ & \Rightarrow \xi(t) &= e^{\lambda t}\xi(0). \end{aligned}$$

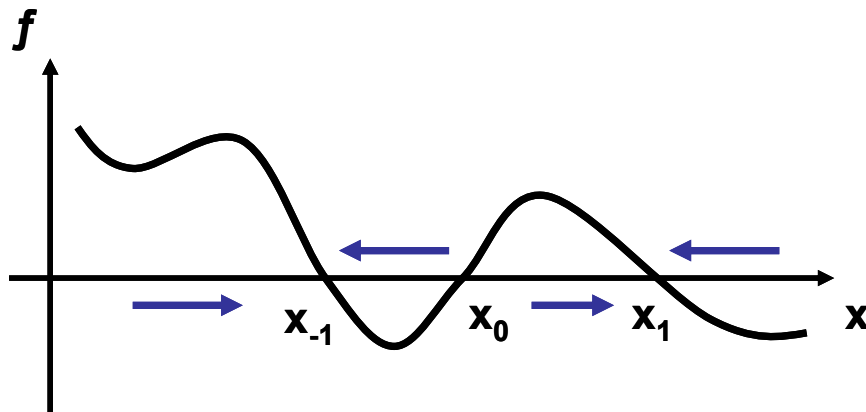
Si $\lambda < 0 \Rightarrow$ le point fixe (PF) est (linéairement) *stable*

Si $\lambda > 0 \Rightarrow$ le PF est " *instable*

Si $\lambda = 0 \Rightarrow$ la stabilité linéaire du PF est *neutre*

Quelques généralités sur les PFs

- 1) $f \in C^1, f \neq 0$ sur tout sous-intervalle : les PFs sont isolés (propriété générique)
- 2) Les bassins d'attraction sont des intervalles ouverts (éventuellement semi-infinis)



2. Bifurcation de col

Comment change-t-elle la géométrie des solutions quand $\mu \neq \mu_0$, ç.a.d. comment changent le nombre et la stabilité des solutions stationnaires ? Nous commençons par l'étude du cas scalaire.

Le cas le plus simple : col ou point-limite

$$\dot{x} = \mu - x^2 \equiv f(x; \mu) \quad (4)$$

$$\text{PFs: } \mu - x^2 = 0$$

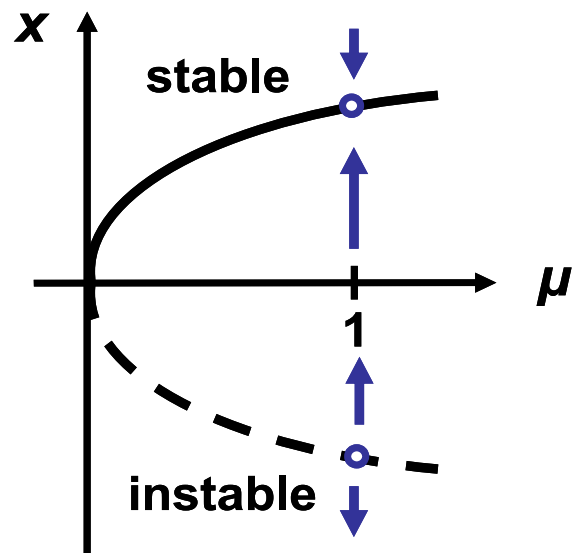
$$x = \pm\sqrt{\mu} \quad (5)$$

Stabilité des PFs :

$$x_1 = \sqrt{\mu}, \quad x_{-1} = -\sqrt{\mu}$$

$$x(0) = x_{\pm 1} + \xi(0)$$

$$\begin{aligned} \dot{\xi} &= \lambda_{\pm} \xi, & \lambda_{\pm} &\equiv f'(x_{\pm 1}) = -2x_{\pm 1} \\ & & &= \mp 2\sqrt{\mu} \end{aligned} \quad (6)$$



∴ Branche supérieure – (linéairement) *stable*

Branche inférieure – (linéairement) *instable*

Nous passons maintenant à l'étude de la stabilité non-linéaire.

3. Bifurcations en n -D

Nous avons étudié le cas scalaire $n = 1$. De façon plus générale, on aura :

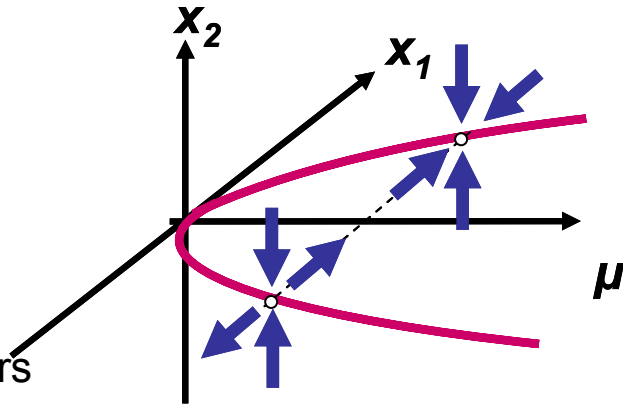
$$\dot{\vec{x}} = \vec{f}(\vec{x}; \mu) \quad , \quad \vec{f} \in C(\mathbb{R}^n \times \mathbb{R})$$

avec $\vec{x} \in \mathbb{R}^n$ et $\mu \in \mathbb{R}$.

Le comportement est “presque” linéaire partout dans l’espace de phase-paramètre $\mathbb{R}^n \times \mathbb{R}$, sauf dans le voisinage de quelques points isolés (x_c, μ_c) : c’est les points de bifurcation, où la matrice jacobienne $L = (\partial f_i / \partial x_j)$ est singulière, c’est-à-dire $\det L = 0$.

Dans le cas $n = 2$, on peut réduire à la *forme normale* :

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2, \\ \dot{x}_2 &= -\lambda x_2, \lambda > 0. \end{aligned}$$



Ce schéma explique d’ailleurs l’appellation “bifurcation de col”.

Dans le cas général, la réduction donne :

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2 \\ \dot{x}_i &= -\lambda_i x_i, \lambda_i > 0, i = 2, \dots, n. \end{aligned}$$

4. Stabilité non-linéaire et principe variationnel

Pour approfondir nos connaissances sur la stabilité, il faut étudier l'effet des perturbations plus importantes.

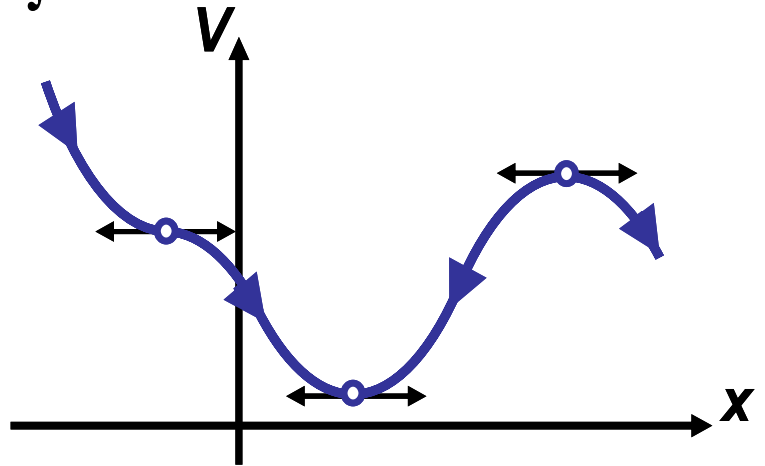
a) Principe variationnel en 0-D

$$\begin{aligned} \dot{x} &= f(x) & V(x) &= -\int^x f(\xi) d\xi \text{ -- pseudo-potentiel} \\ &= -V'(x) \\ \dot{x}^2 &= -\frac{dV}{dx} \frac{dx}{dt} = -\dot{V} \end{aligned}$$

$\therefore V$ va décroître le long de la trajectoire de l'EDO tant que

$$\dot{x} \neq 0 \Leftrightarrow V' \neq 0$$

$\dot{x} = 0$ si V atteint un minimum, maximum ou col.

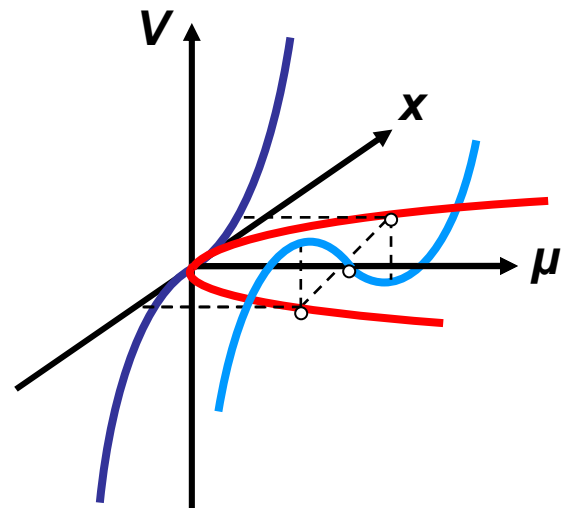


Evidemment, seulement $V = \text{min}$ est *stable* – de façon non-linéaire.

Avec ce résultat, nous retournons à la bifurcation de col.

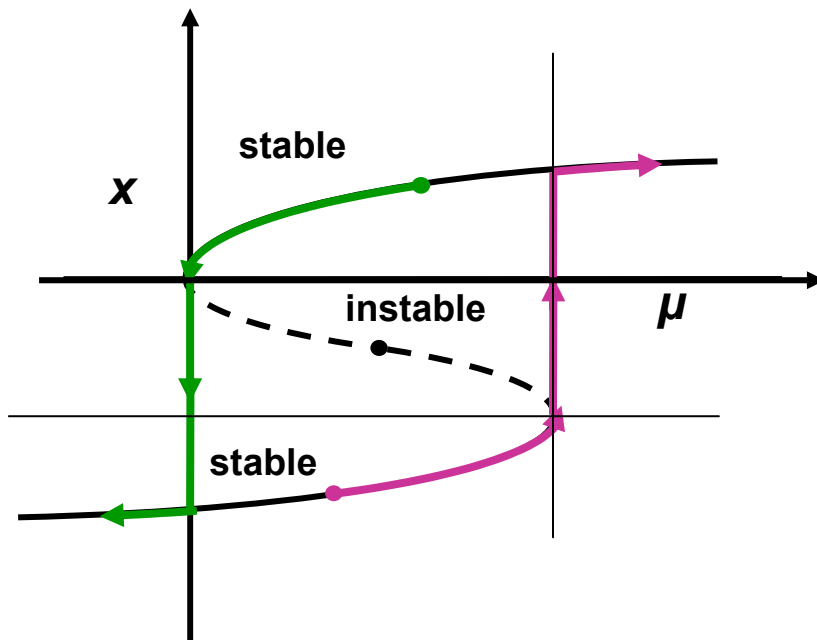
$$\dot{x} = \mu - x^2,$$

$$V(x; \mu) = -\mu x + x^3 / 3 + c(\mu).$$



5. Bistabilité et hystérésis

L'association de deux bifurcations de col peut créer un phénomène d'hystérésis (courbe en forme de S) :

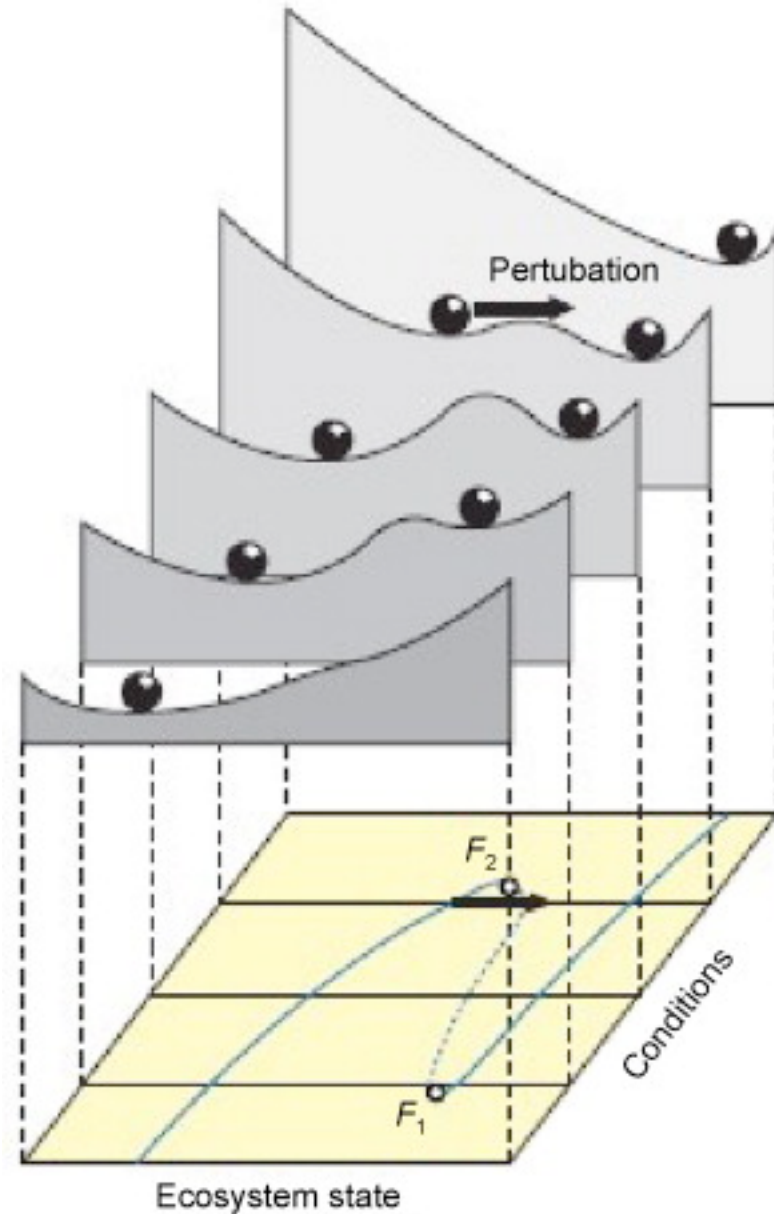


$\dot{x} = \mu - x^2$: la bifurcation en haut à gauche

$\dot{x} = (\mu - 1) + (x + \frac{1}{2})^2$: la bifurcation en bas à droite

“Bifurcation pli” en version visuelle

- Plotter la position de l'équilibre en fonction du paramètre
- La courbe est pliée



Double-well potential in 2-D

1-D EBM of Budyko-Sellers-North, cf. Held & Suarez (*Tellus*, 1974); North *et al.* (*JAS*, 1979).

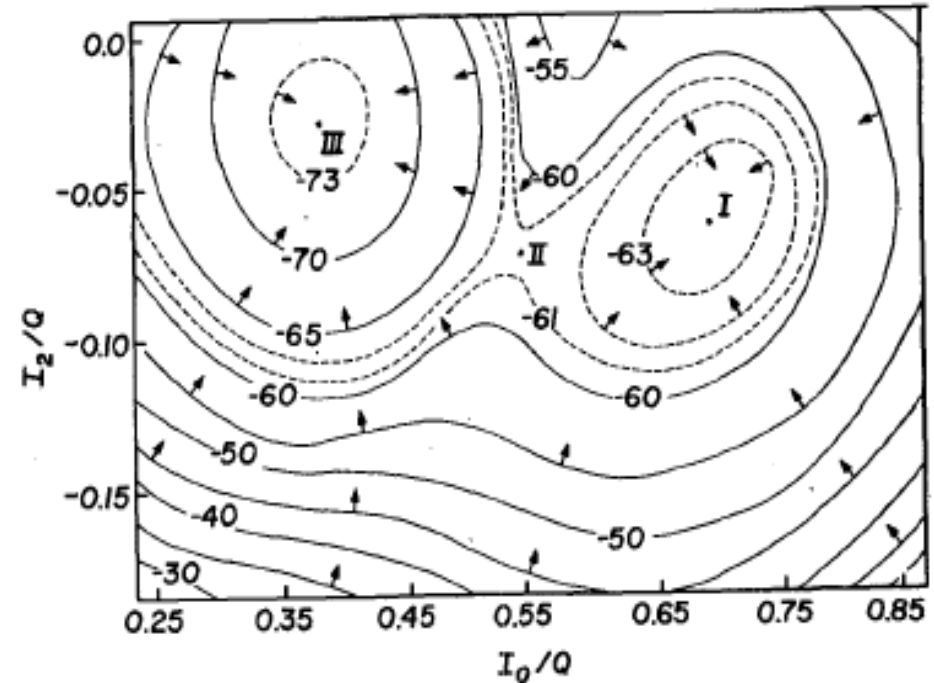
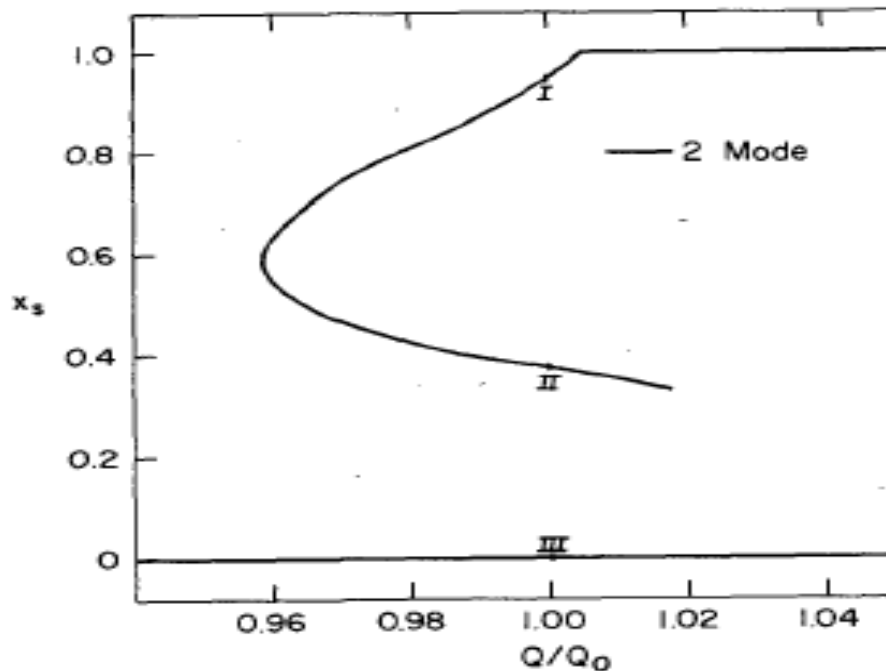
Taking $x = \sin(\text{latitude})$ and $k(x, T) = k_0$,
we get the semi-linear parabolic PDE

$$CT_t = [k_0(1 - x^2)T_x]_x + Q(x)[1 - \alpha(T)] - I(T)$$

which yields the variational principle:

$$F\{T(x)\} = \int \left\{ \left[\frac{1}{2} k_0 (1 - x^2) T_x^2 - Q(x) A(T) + R(T) \right] dx \right\}, \text{ where}$$

$$A(T) = \int^T [1 - \alpha(T)] dT, \text{ and } R(T) = \int^T I(T) dT.$$



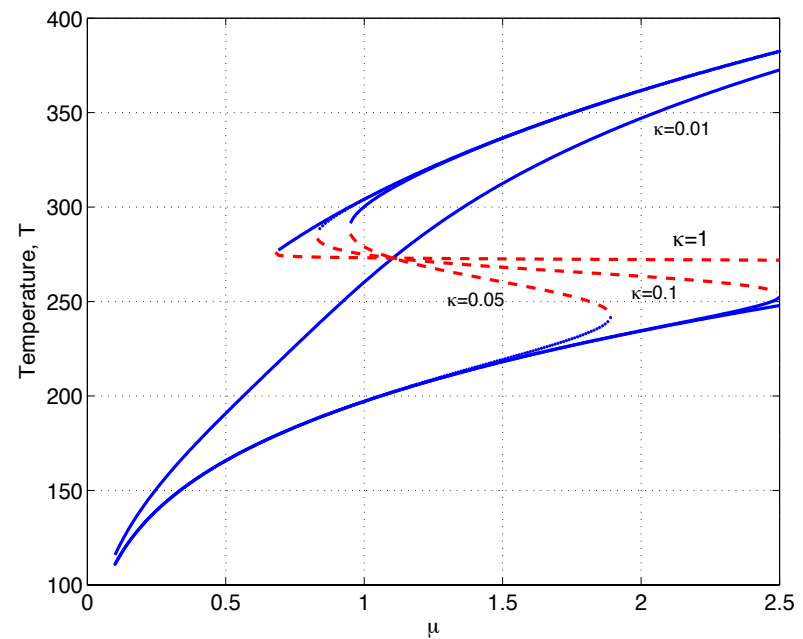
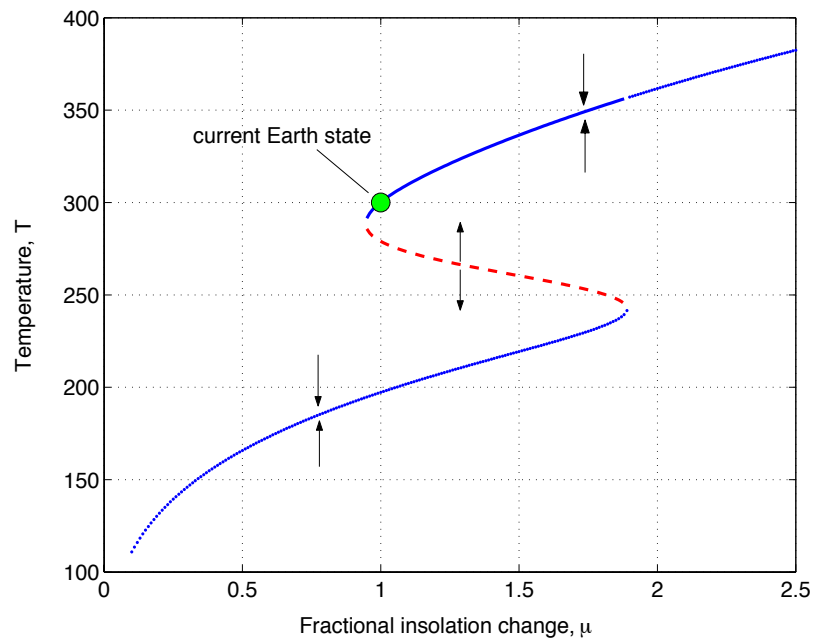
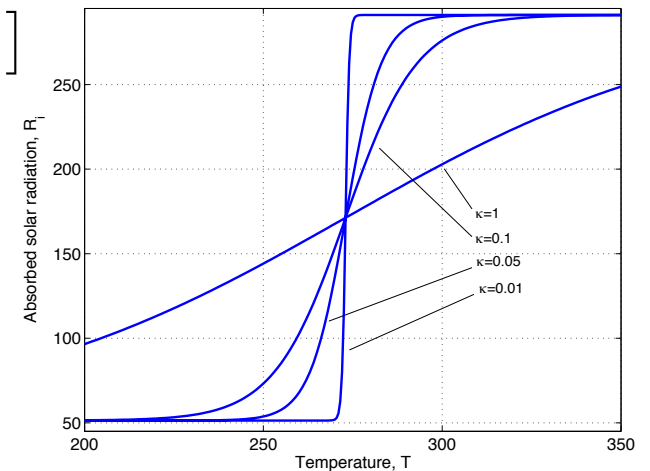
Distance to “tipping points”?

Slightly modified 0-D EBM (Zaliapin & Ghil, *NPG*, 2010)

$$c\dot{T} = \mu Q_0 (1 - \alpha(T)) \sigma T^4 [1 - m \tanh((T/T_0)^6)]$$

$$\alpha(T; \kappa) = c_1 + c_2 \frac{1 - \tanh[\kappa(T - T_c)]}{2}$$

T_c is the ice-margin temperature,
while κ is an extra “Budyko-vs.-Sellers” parameter



IMAGES SIMPLIFIEES DE LA CIRCULATION MOYENNE

Circulation directe de Hadley

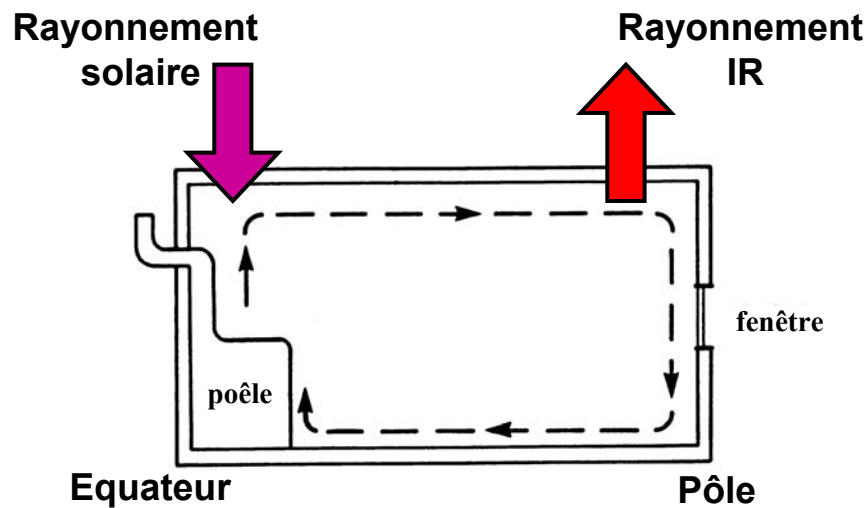


Image idéalisée de la circulation globale de l'atmosphère. *

Circulation observée

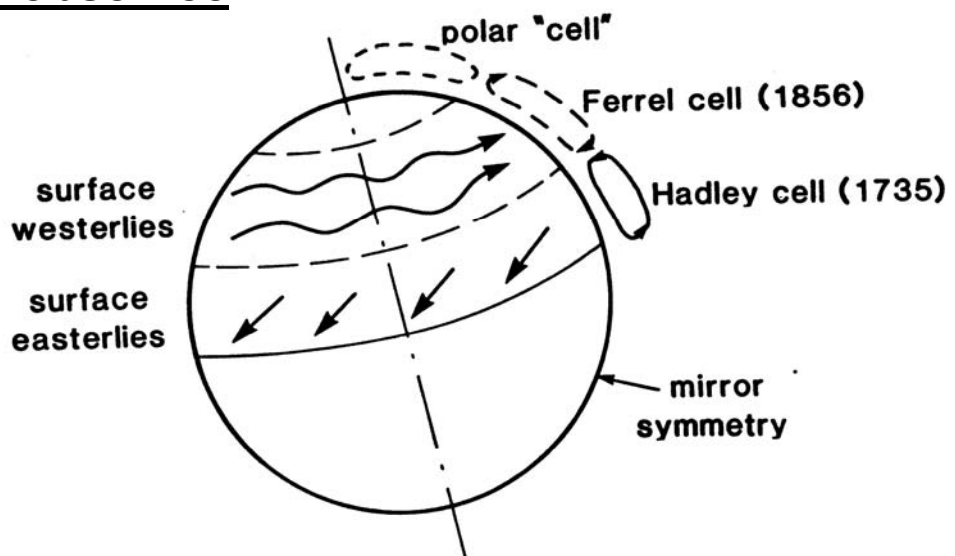


Schéma de la circulation générale de l'atmosphère. *

* D'après Ghil and Childress, Chap. 4

Modèles de bilan énergétique

(Energy-balance models, EBMs)

$$C \frac{\partial T}{\partial t} = R_i - R_o + D$$

C — capacité calorifique locale

T — température de surface locale

R_i — rayonnement solaire incident

R_o — rayonnement terrestre vers l'espace

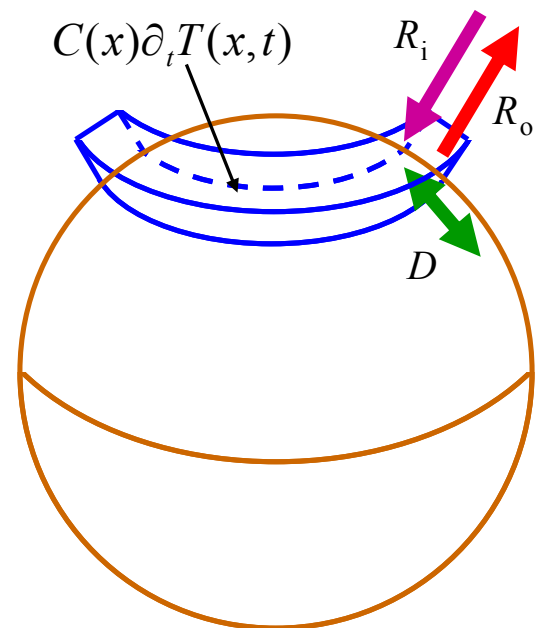
D — redistribution de la chaleur (“diffusion”)

Remarques :

1. une variable d'espace maximum (x)
2. C , R_i , R_o et D doivent être calculés (“paramétrés”) en fonction de $T=T(x,t)$
3. Principale caractéristique du modèle :

$$R_i = Q(x) \{1 - \alpha(x, T)\}$$

avec α l'albédo local



Energy-balance models (EBMs)

Version O-D (moyennée sur le globe)

$$\begin{aligned} C \frac{d\bar{T}}{dt} &= R_i - R_o \\ &= Q \{1 - \alpha(\bar{T})\} - \sigma \bar{T}^4 m(\bar{T}) \end{aligned}$$

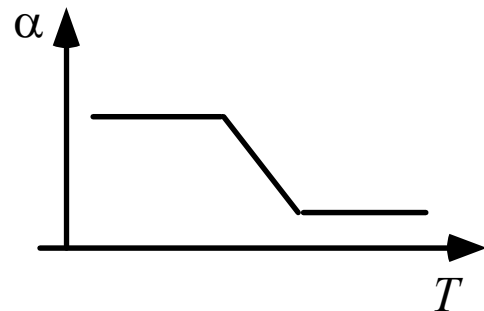
\bar{T} — température moyenne à la surface

t — temps (en milliers d'années)

Q — flux solaire incident

α — albédo

C — capacité calorifique



σ — constante de Stefan–Boltzmann

m — facteur d'effet de serre

Remarque: α est fonction de l'extension de la glace et de la neige, de la nébulosité, etc. (variables implicites), tout étant paramétré en fonction de la variable explicite \bar{T} .

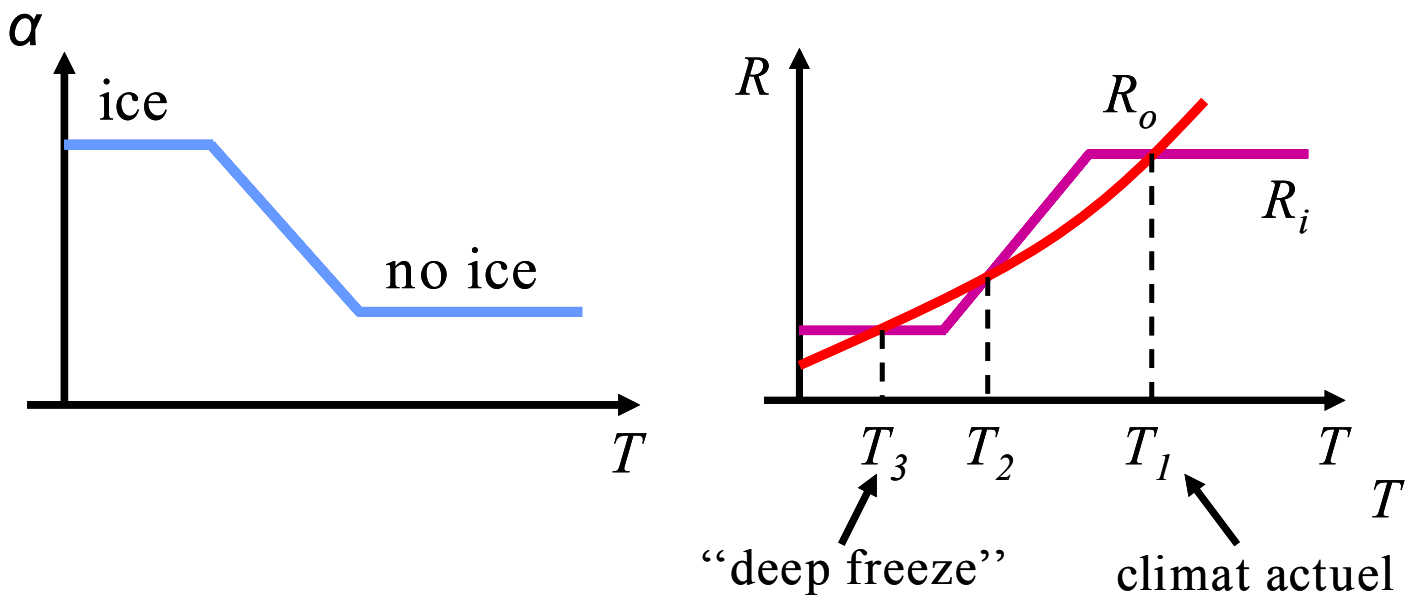
Solutions du modèle

On veut écrire T sous la forme:

$$T = T(t; T_0, Q, c, \dots)$$

Solutions stationnaires :

$$Q\{1 - \alpha(T)\} - \sigma T^4 = 0$$



Que se passe-t-il si le soleil « clignote » et $T = T_1 + \Delta T$?

Il faut reprendre l'équation d'origine, qui dépend du temps.

Condition de stabilité

$$C \partial_t T = R_i - R_o = f(T)$$

$$R_i = Q \{1 - \alpha(T)\}$$

$$R_o = A + BT$$

On pose $T = T_j + \theta$:

$$f(T_j) = 0,$$

$$f(T) = f(T_j) + f'(T_j)\theta + \dots$$

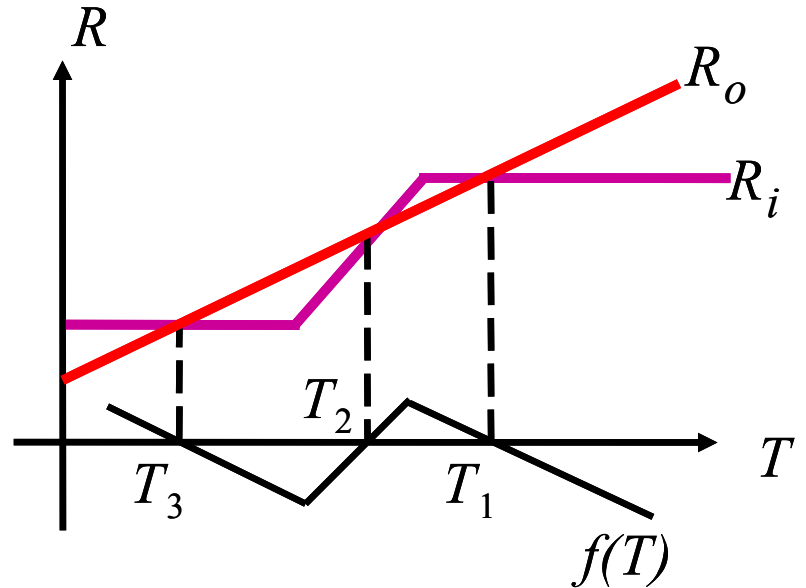
Soit $\lambda_j \equiv f'(T_j)/c$

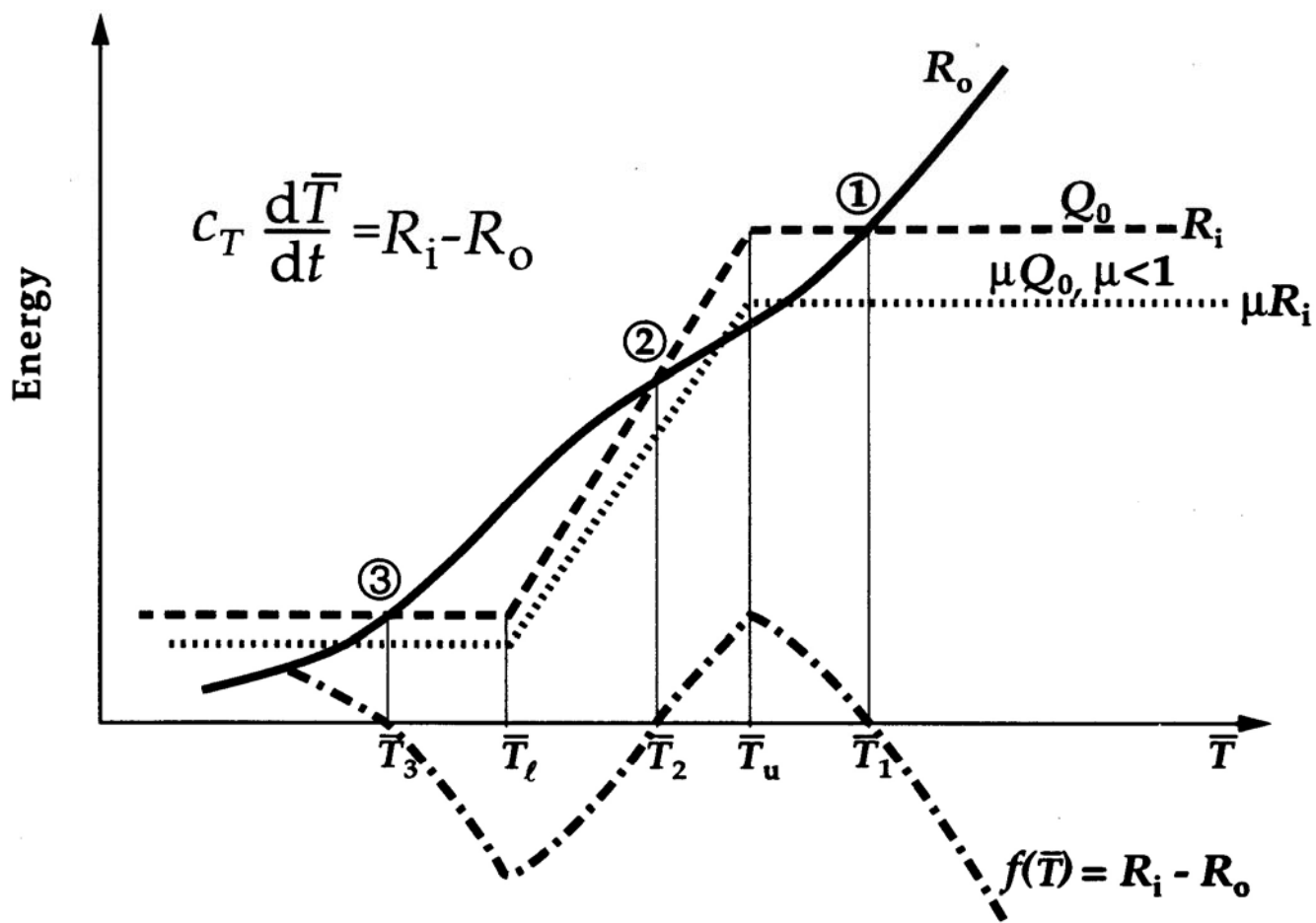
$$\partial_t \theta = \lambda_j \theta \Rightarrow \theta = e^{\lambda_j t} \theta_0.$$

Si $\lambda_j < 0$ stable

Si $\lambda_j > 0$ instable

Remarque: dans le cas 1-D, $\lambda_j \rightarrow \lambda_j^{(0)}$; $\lambda_j \sim 1/c$





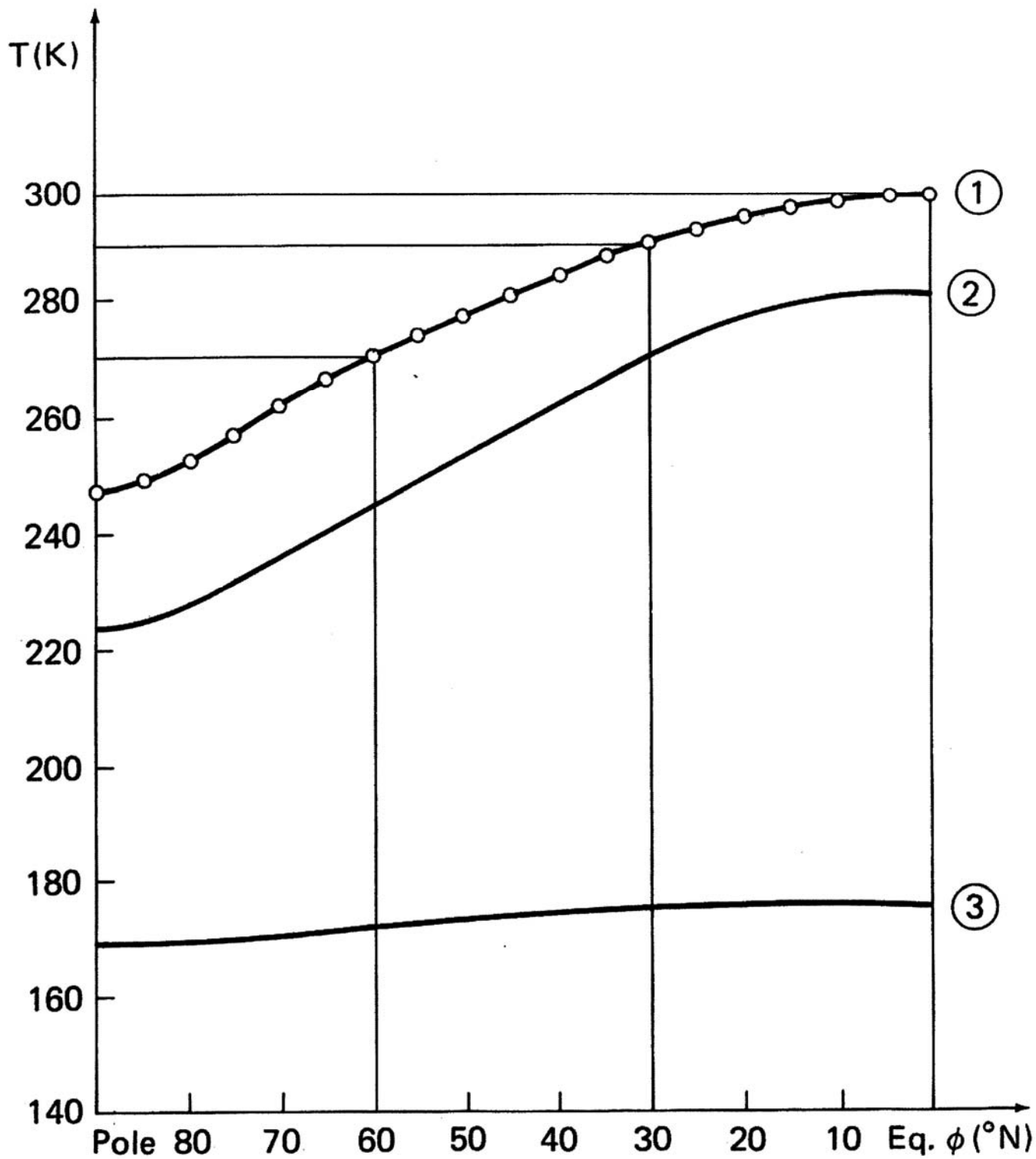
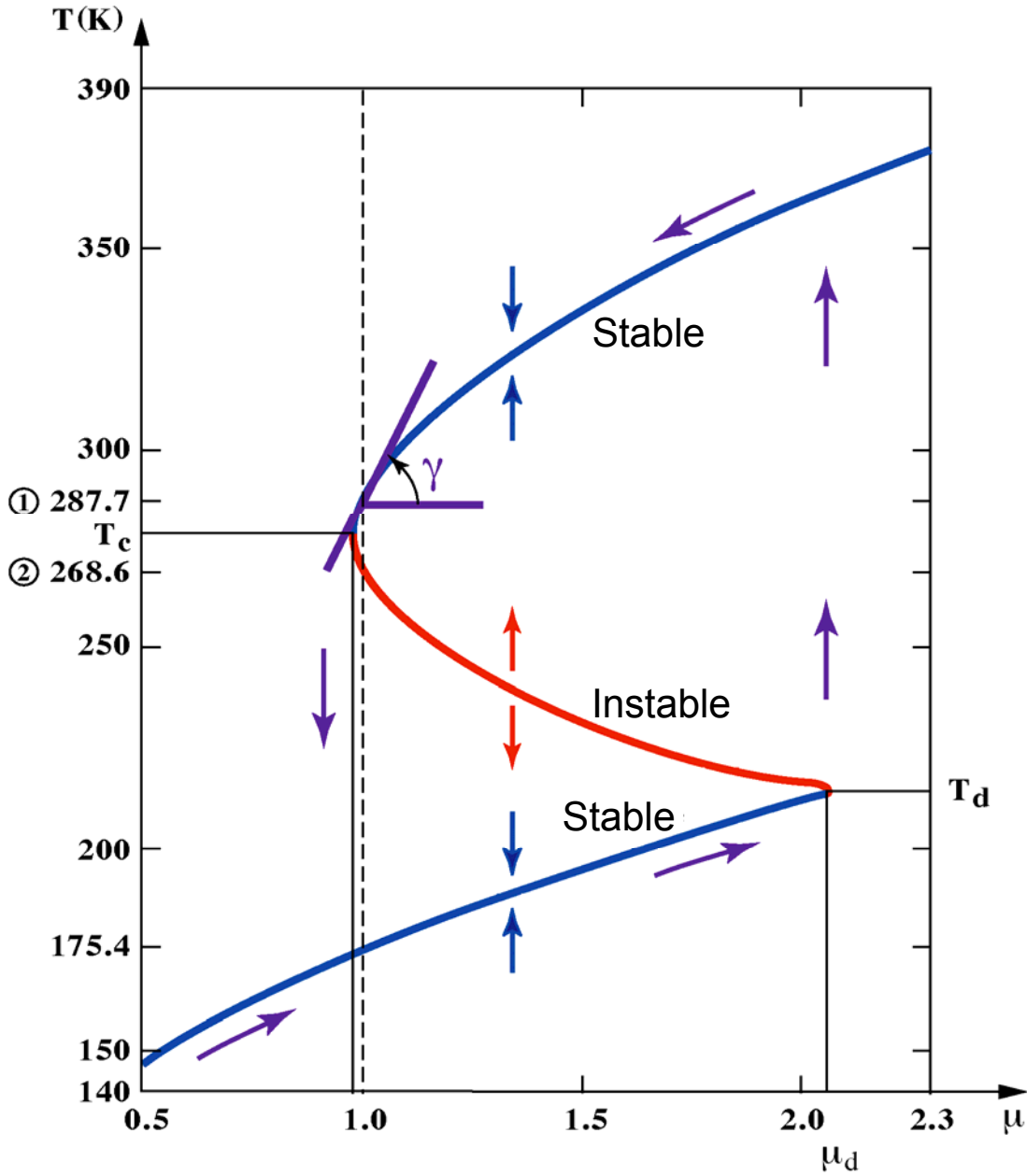


Diagramme de Bifurcation

EBM 1-D: $C(x)T_t = \{k(x,T)T_x\}_x + \mu Q_0 \{1 - \alpha(x,T)\} - g(T)\sigma T^4$
 $T_x = 0$ at $x = 0,1$



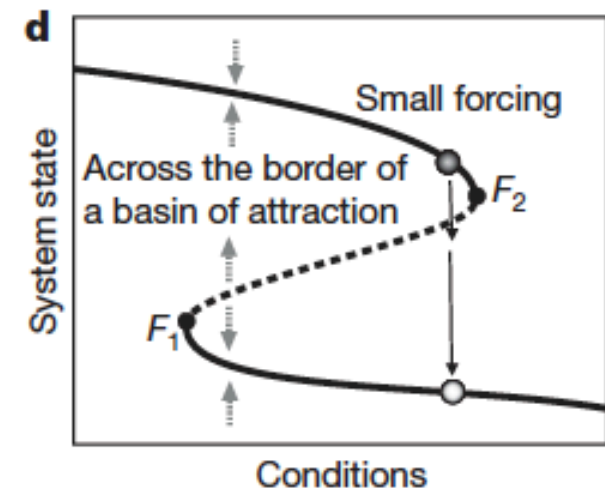
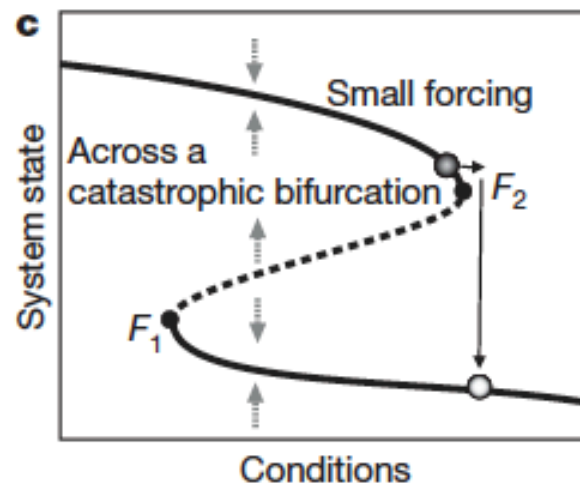
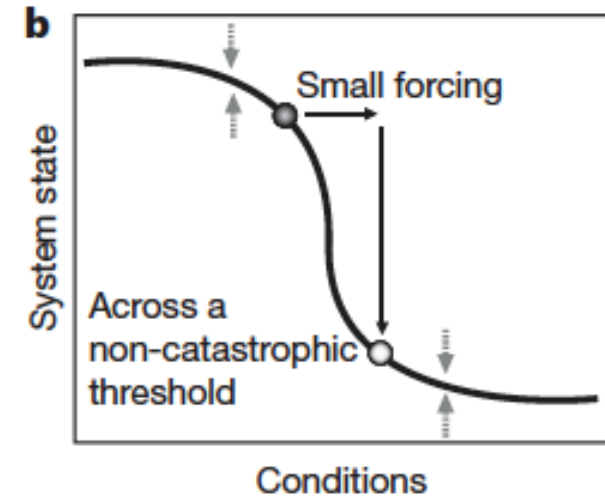
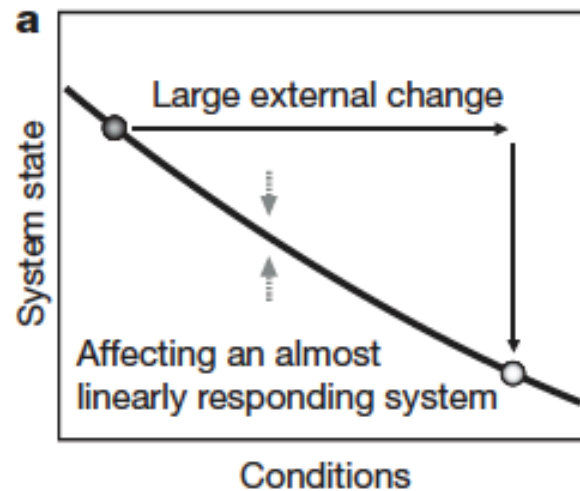
Sensibilité du climat: $\gamma = \frac{dT}{d\mu} \cong 0,01$ (1K par % de Q)

Quelques conclusions

- ◆ “Tipping points” (“points de non retour” ou “points de basculement”) et bifurcations: **équilibres multiples** et **transitions rapides** entre ceux-ci.
- ◆ **Prévision** de ces transitions? A suivre ...
- ◆ **Transitions** entre types de comportement plus généraux — **cycles limite, attracteurs étranges** — également à suivre ...

Prévision d'un point de non retour ? - I

Réponse "lisse" ou
réponse "abrupte"

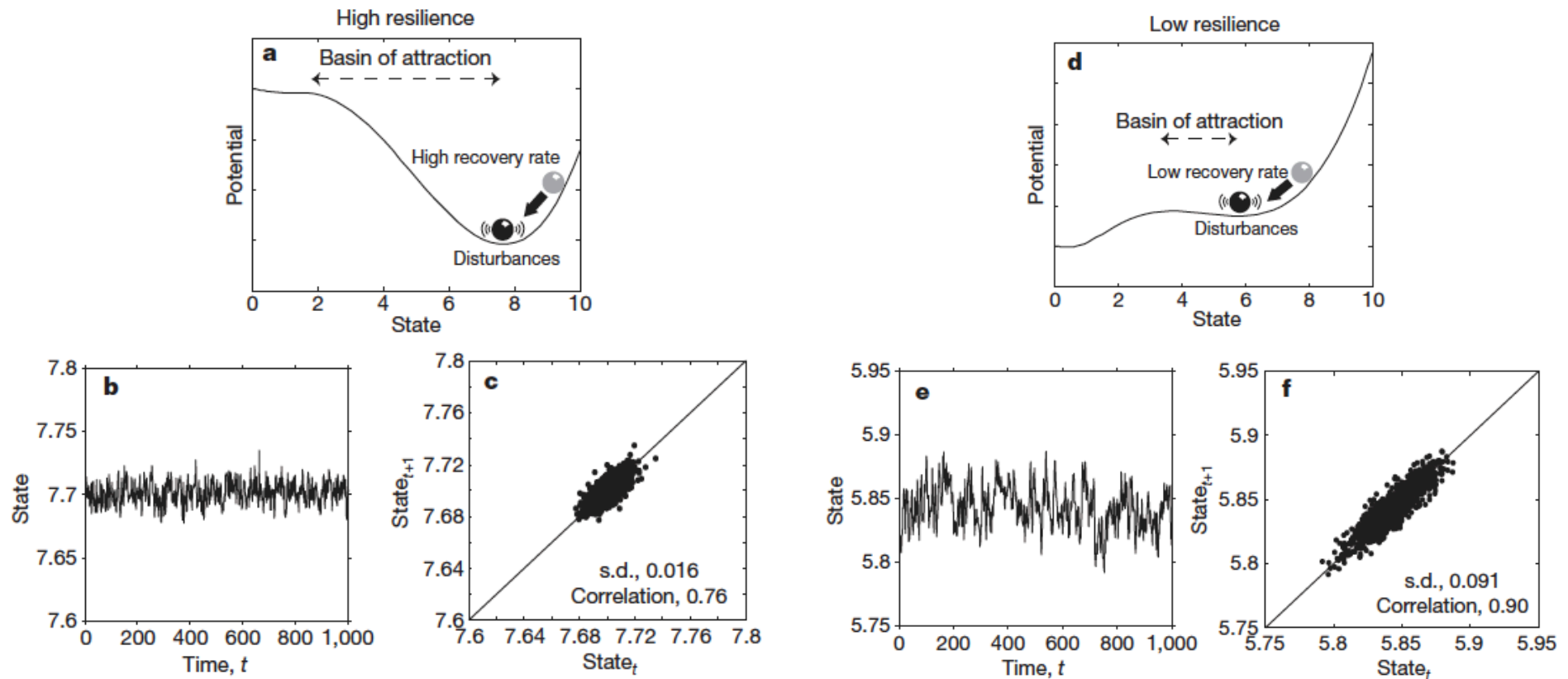


Prévision d'un point de non retour ? - II

“Résilience”

et

labilité



M. Scheffer *et al.*, Early warning ... (*Nature*, 2009)

Un peu de bibliographie

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Reserve slides

Modèle 1-D (EBM "classique")

$$(1) C(x)T_t = R_i - R_o + D$$

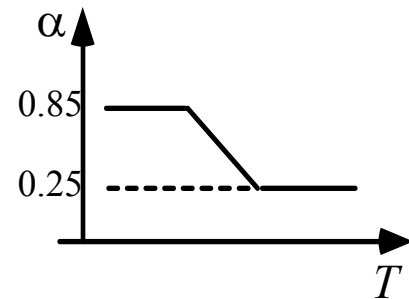
T – température

x – coordonnée latitudinale, $0 \leq x \leq 1$

$$R_i = Q(x)\{1 - \alpha\}$$

$$= Q(x)\{1 - b(x) + c_1 T\}_c$$

$$R_o = \sigma T^4 \{1 - m \tanh(c_3 T^6)\}$$



$\tilde{T}(x)$ – le climat observé

$$D = \frac{1}{\sin \frac{\pi x}{2}} \partial_x \sin \frac{\pi x}{2} \{k(x) + k_s(x)g(\tilde{T})\} T_x$$

$$\text{C.B. : } T_x(0) = T_x(1) = 0$$

$x = 0$ Pôle (Nord)

$x = 1$ Equateur

Questions: 1. Solutions stationnaires ("climats") ?

2. Stabilité ?

3. Perturbation & bifurcation ?

$$Q \rightarrow \mu Q \quad (\mu = 1)$$