Scale free distributions

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The Great Wave of Kanagawa, Hokusai 1831.

The central question "Why does one so often observes secular waves ?"

The answer relates to scale free distributions.

Scale free distributions are probability distributions lacking the property of a characteristic element which would give the order of magnitude of the other elements.

In other words, they lack a characteristic scale.

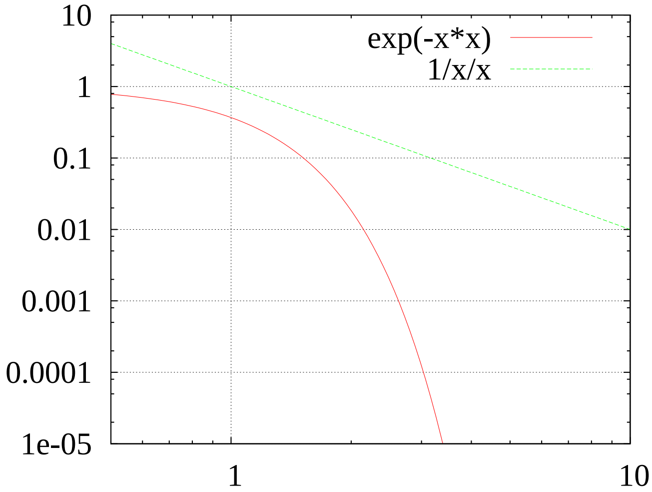


Figure 1: Loglog plot of Gaussian and scale free distributions.

We are familiar with normal distribution such as body size of humans, span of life time, planes or cars sizes, records of athletes etc.

By contrast city sizes in population or area, wealth in capital or revenues, rivers, earthquakes, waves, returns in the stock exchange have scale free distributions: the sizes are distributed on several orders of magnitude.

Because we are more familiar with normal distributions, we tend to expect all distributions to be "normal" and are surprised by the occurence of extreme events: if the distribution of all events were normal, extreme events out of the normal range would be extremely rare as predicted by the Gaussian law of probability. Scale free distributions display extreme event rarely, but with a higher frequency than normal distributions, thus surprising us.

1 Power law distributions: properties

The most common example of a scale free distribution is the power law:

$$P(x) \sim x^{-\alpha}$$

with positive α , which graph in log-log coordinates is a straight line.

For instance, Gutenberg Wagner law (1944) is written:

$$logN(M) = a - bM$$

where $N(M)$ is the frequency of earthquakes with magnitude M. The magnitude M is defined as the logarithm of the ratio of the amplitude of waves measured by a seismograph to an arbitrary small amplitude. An earthquake that measures 5.0 on the Richter scale has a shaking amplitude 10 times larger than one that measures 4.0, and corresponds to a 31.6 times larger release of energy. Gutenberg Wagner law is then a power law relating the frequency and the energy released during earthquakes. As contrast to the succession of momenta of bell-shaped distributions, higher momenta of power law distribution diverge, as directly observed from their expression:

$$M(m) = \int_0^\infty x^m x^{-\alpha} dx \tag{1}$$

Momenta are finite for

$$m < \alpha - 1 \tag{2}$$

For $\alpha \leq 1$, even averages are not defined, and for $\alpha \leq 2$ standard deviation are not defined. Of course in practice, when a finite set of empirical data is available, any momentum can be computed; but its value might not give much insight.

Furthermore, when it comes to empirical situations, a power law distribution is not applicable from 0 to ∞. Limits in measurement accuracy and the size of the system e.g. restrict the range of the distribution.

After a brief history of the subject, I will described two different mechanisms at the origin of scale free distributions.

2 History

Pareto (1896) was interested in the distribution of income: how many people have an income greater than x. Pareto's law is given in terms of the cumulative distribution function (CDF), i.e. the number of events larger than x is an inverse power of x:

$$P[X > x] \sim x^{-k} \tag{3}$$

$$P[X = x] \sim x^{-(k+1)} \tag{4}$$

It states that there are a few multi-billionaires, but most people make only a modest income.

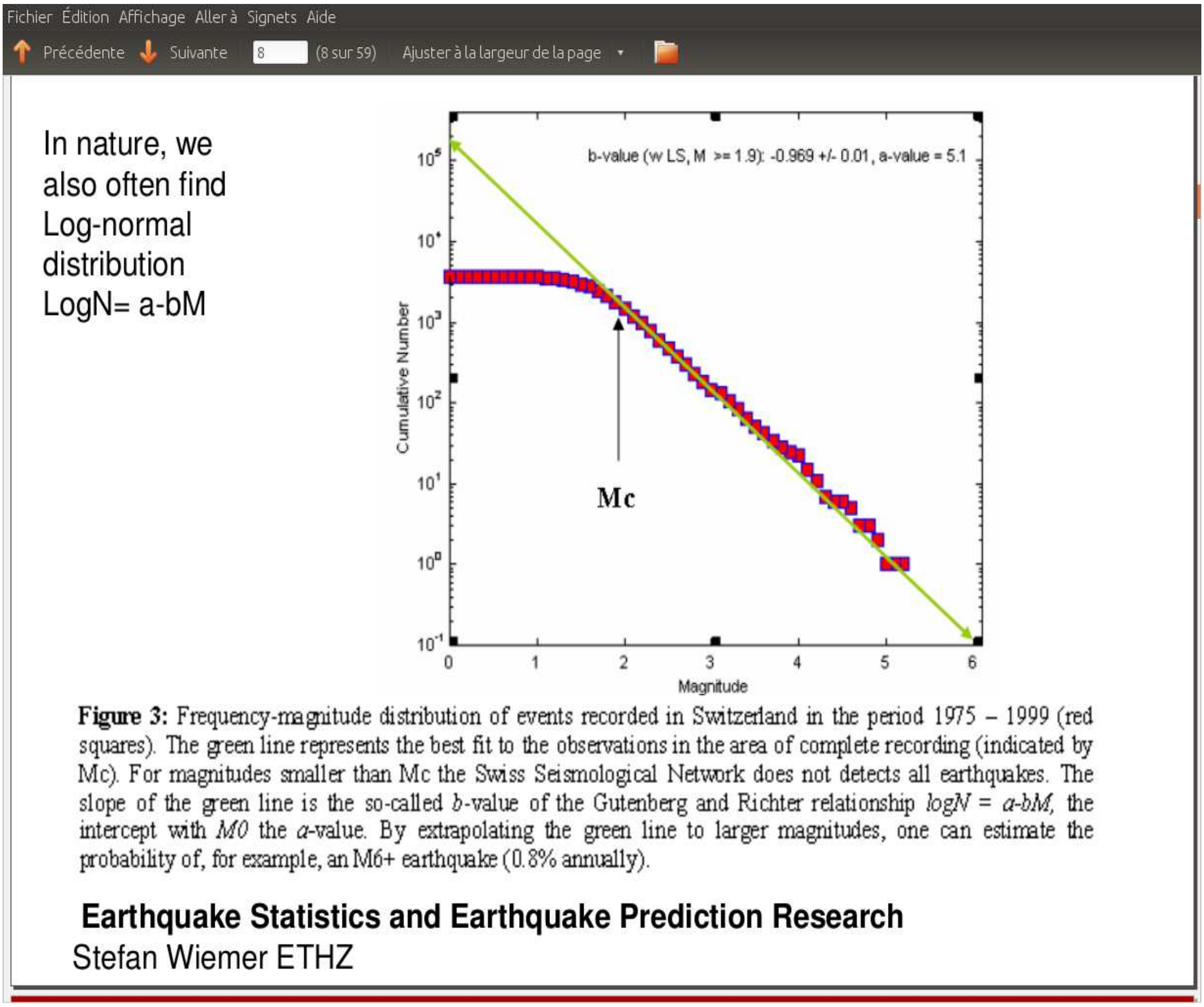
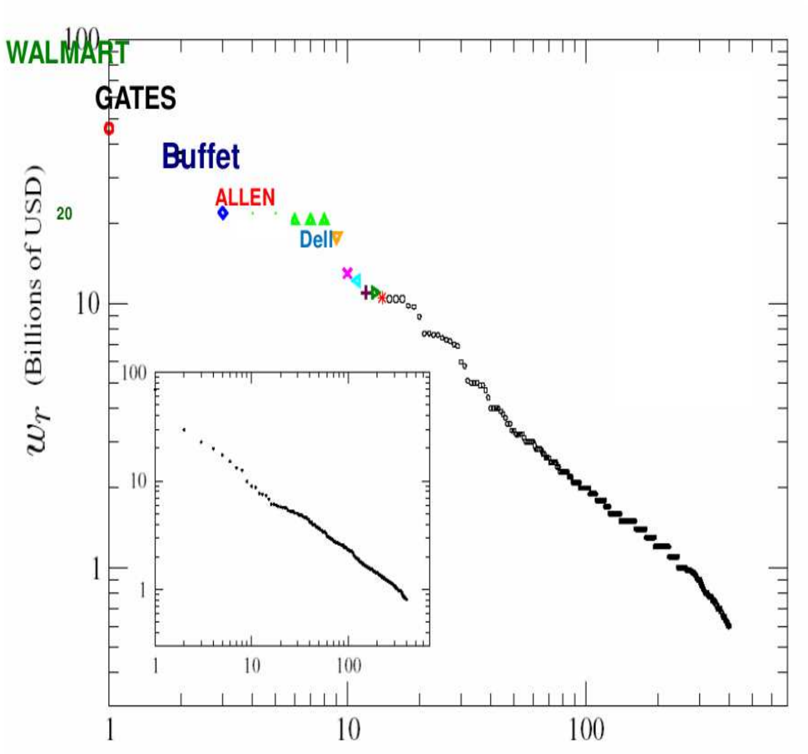


Figure 2: Gutenberg Wagner plot of earthquakes in Switzerland. Note the deviation from the power law at small magnitudes

Zipf plot of the wealths of the investors in the Forbes 400 of 2003 vs their rank



Zipf's law (1932) states that given some corpus of natural language utterances, the frequency f of any word is inversely proportional to its rank r in the frequency table.

$$f \sim r^{-b} \tag{5}$$

with b close to unity.

3 Interactions

PHYSICAL REVIEW LETTERS

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Self-Organized Criticality: An Explanation of 1/f Noise

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We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or 1/f noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

Bak proposes a cellular automata model of avalanches in sand piles with universal properties. We take a two dimension lattice of identical cell. The state of a cell represents the relative height of sand grains, 0,1,2,3 .. on that site. Grains of sand are randomly pouring on the cells. When a site height is well above its neighbours, 4 grains of sand are redistributed to its nearest neighbours. This formally translates as each time the state is larger than a threshold k, the states of cell i, j and its neighbour are updated according to:

$$s(i,j) = s(i,j) - 4 \tag{6}$$

$$s(i \pm 1, j) = s(i\pm, j) + 1 \tag{7}$$

$$s(i, j \pm 1) = s(i, j \pm 1) + 1 \tag{8}$$

Since the neighbours height are increased, they might also reach a level above the threshold, and loose 4 grains themselves. The process is carried on until no other cell is above threshold. This succession of events triggered by locally adding only one grain on a site is an avalanche, which size S is the total number of cells reached by the avalanche.

Demo: Sandpile NetLogo http://ccl.northwestern.edu/netlogo/models/Sandpile

Bak sandpile model is based on the transfer and diffusion of stress to neighbouring cells until the boundary remains stable.

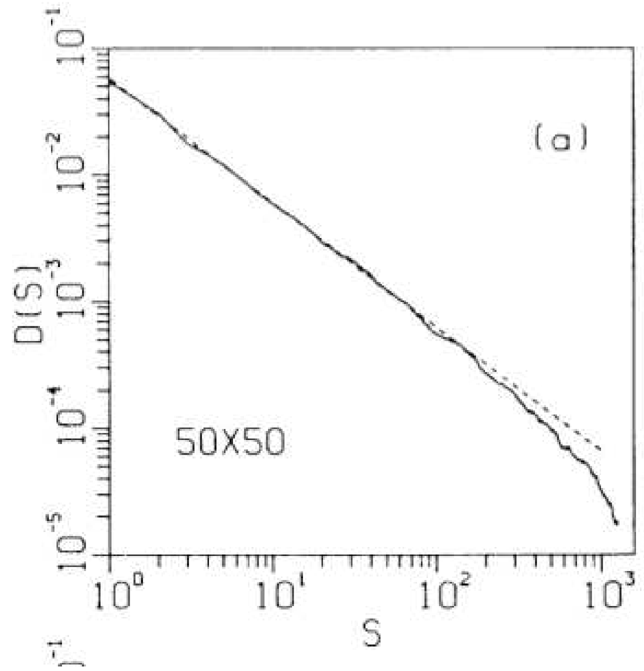

Figure 3: Simulation results: the frequency of avalanches of size S obeys a scale free distribution as shown by the straight line of the log-log plot

4 Multiplicative processes

A random multiplicative process yields a log-normal distribution. If we e.g. assume that profits of firms are proportional to their capital, capital increase K_i of firm i obeys:

$$\frac{dK_i}{dt} = \lambda_i(t) K_i(t) \tag{9}$$

where $\lambda_i(t)$ is a noisy multiplicative term specific to firm i. The noise reflects the aleas of economic profit. The equation can be rewritten:

$$\frac{dlog(K_i)}{dt} = \lambda_i(t) \tag{10}$$

Since $log(K_i)$ describes a random walk, its distribution is Gaussian. The distribution of K_i is thus lognormal and much wider than a Gaussian distribution.

A scale free distribution is obtained when one also imposes a lower boundary to K_1. Here follows a heuristic illustration. Let us distribute wealth levels in bins located at w_i such that:

$$w_i = 2w_{i-1} \tag{11}$$

The constant factor 2 comes from the *multiplicative* process. One possible realisation of a *random* multiplicative process is to suppose that any given time, the probability to increase wealth (going to the right from bin w_i to bin w_{i+1}), is $\frac{1}{k}$ the probability to decrease it to bin w_{i-1}. The following figure illustrate the process: Stability is achieved at any boundary between bins i and $i+1$ when the flow to the right (from bin i) equals the flow to the left (from bin $i+1$), thus when:

$$N(w_i) = kN(w_{i+1}) \tag{12}$$

where $N(w_i)$ is the population of agents with wealth i. When we iterates from level i to level 1 where no left transitions are allowed we obtain:

$$N(w_i = 2^i w_1) = (k)^{-i} N(w_1) \tag{13}$$

$$log(N(2^i w_1)) = log(N(w_1)) - \alpha.i \tag{14}$$

Where $\alpha = log(k)$. The last equation shows that the population and wealth have a linear dependance on a log scale (QOD).

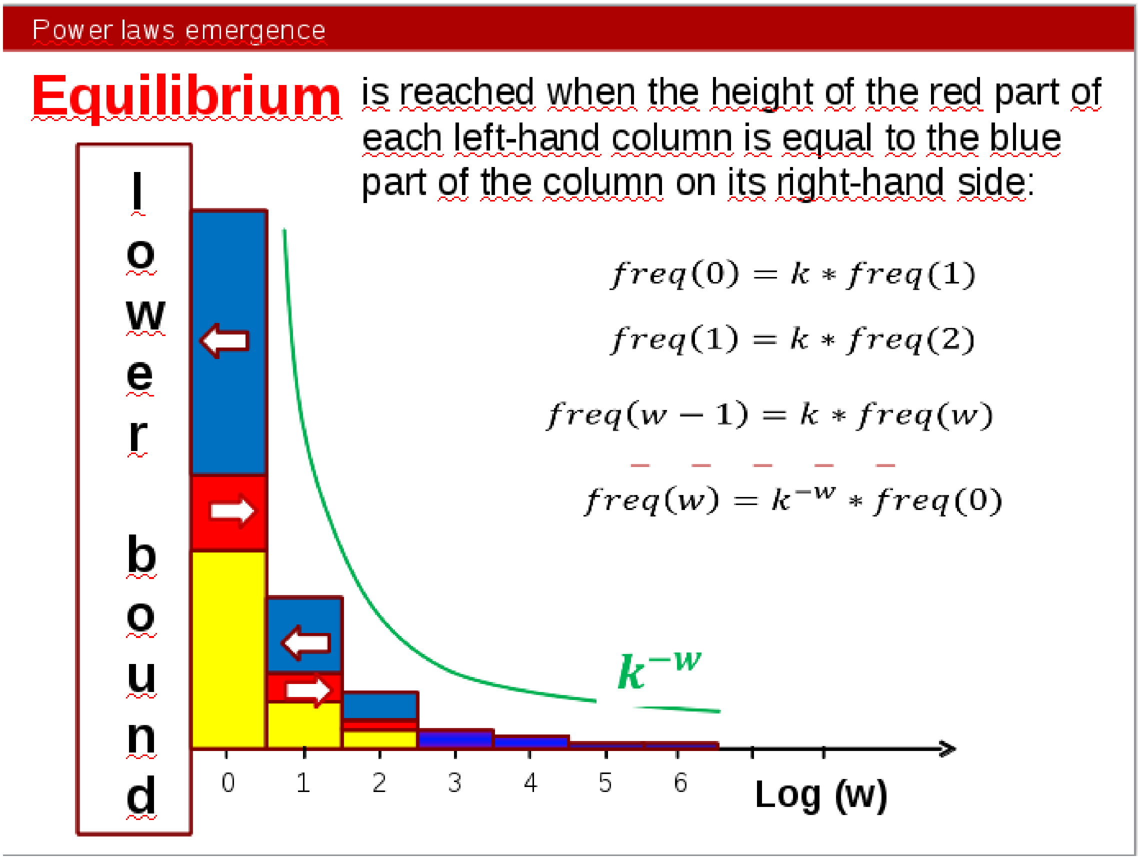

Figure 4: Exchanges between wealth bins. The color code is: yellow don't move, blue wealth decreases, red wealth increases.

This computation and the figure illustrate the similarity between the Bak avalanche process and the random multiplicative process.

The random multiplicative process applies to distribution of wealth, revenues, firm and city sizes etc.

5 Environment

Because of stress propagation, scale invariant distributions are pervasive in environmental risks:

- Physical processes (Earthquakes, waves, tsunamis, avalanches, floods)
- Biological processes (mass extinction in trophic networks, Bak Sneppen)
- Consequences of catastrophes: power grid blackout, consequences of extreme climatic conditions such as blizzard on air, road and train traffic
- and even in rescue (interrupted communications) and recovery (propagation of shortages across production networks)

The above examples illustrate a notion of systemic risk, i.e. large consequences due to the propagation of failures across a network of connected elements.

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